ON CONVERGENCE AND ORDER OF A NUMERICAL INTEGRATOR FOR SOLVING LINEAR STIFF FIRST ORDER INITIAL VALUE PROBLEM

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Abstract

Solution to stiff initial value problems had been studied by authors, [6], [2], [3] and [4]. In the light of this, [5] proposed a new numerical integrator to cope with linear stiff first order initial value problems with constant matrix of order 2.

As an extension of the work done in [5], we studied the order and convergence of the integrator. The integrator is of order 5 and the rate of convergence would be high for a very small meshsize, h.

Introduction

Stiff initial value problems were first encountered in the study of the matrix of springs of varying stiffness.

Some researchers, [6], [10], [9], [8], and [7] had contributed immensely towards the solution of such differential Equations.

Definition 1.1: STOER AND BULIRSCH [1 2]

The function f(x, y) is said to satisfy a Lipschitz condition of order one with respect to y in the domain, D if there exists a constant $k > 0$ such that

$$|f(x, y) - f(x, z)| \leq \kappa \, |y - z| \qquad (1.1.1)$$

for all (x,y) and (x ,z) in D.

Let us now consider the linear stiff first order initial value problem with constant matrix, A* of order 2

$$y' = f(x,y) = A^* y, \quad y(a) = \beta \qquad (1.1.2)$$

where

$$y = \{y_1\} \quad \text{and} \quad \beta = \{\beta_1\}$$
$$\{y_2\} \text{ and } \beta = \{\beta_2\} \qquad (1.1.3)$$

It is assumed in [2] that the function f(x, y) is defined and continuous in the region $\Gamma = R \times R^2$ where $R = a \leq x \leq b$ is a finite interval on the real line and $y \in R^2$. In addition, the function f(x, y) satisfies a Lipschitz condition of order one with respect to y.

Definition 1.2. LAMBERT [3]

The initial value problem (1.1.2) is said to be stiff over the interval $R = a \leq x \leq b$ if for every x in the interval the eigenvalues, $\{\lambda_i / i = 1,2\}$ of A* satisfy the following conditions:

(a) $\operatorname{Re}(\lambda_i) < 0$, for $i = 1,2$

(b) $\operatorname{Max} \left| \dfrac{\mu_i}{\nu_i} \right| >> 1$

with $(\lambda_i = \mu_i \pm i\nu_i) / i = 1,2 \}$

Numerical Integrator For Solving
Linear Stiff Initial Value Problem

If y_n denotes the numerical approximation to the exact solution y (x,,) at $x = x_n$ then, adopting the interpolating function

$$F(x_n) = A \exp(\lambda_1 x_n) + B \exp(\lambda 2\, x_n) + c \quad (2.1)$$

[5] proposed the numerical integrator

$$y_{n+1} = y_n + G(h), (h) f \times H(h), f_n^1, \quad n = o, 1, 2, \ldots\ldots.. \quad (2.2)$$

where the quantities G (h), H (h), f_n and f_n^1 are defined as:

$$f_n = \lambda_1 \exp(\lambda_1 x_n) + B \lambda_2 \exp(\lambda_2 x_n) \quad (2.3)$$

$$f_n^1 = A \lambda_1^2 \exp(\lambda_1 x_n) + B \lambda_2^2 \exp(\lambda_2 x_n) \quad (2.4)$$

$$G(h) = \frac{(\mu^2 - \nu^2) \mathrm{Sin}(\nu h) \exp(\mu h) - 2\mu\nu \exp(\mu h) \cos(\nu h) + 2\mu\nu}{-\nu(\mu^2 - \nu^2)} \quad (2.5)$$

$$H(h) = \frac{\nu \exp(\mu h) \cos - 2\mu \exp(\mu h) \sin \nu h - \nu}{-\nu(\mu^2 + \nu^2)} \quad (2.6)$$

and $h = x_{n+1} - x_n$, with $h \, \mathcal{E} \, (0, 1]$ (2.7)

the authors, [1] and [4] suggested the following desirable constraints on (1.1.2) and (2.1)

$$y_{n+j} = F^1(x_{n+j}), \qquad j=0,1 \quad (2.8)$$

$$f_n = F^1(x_n) \quad (2.9)$$

3.0: Convergence Of The Numerical Integrator

We shall investigate the convergence of the numerical integrator by adopting the computer Algorithm proposed in [5].

Definition **3.1: Hairer, Norsett And Wanner [11]**

Let y_n be the numerical approximate solution to a linear stiff first order initial value problem, and y(x,,), the theoretical solution to the initial value problem.

If $y_n \to y(x_n)$ as the number of iteration, n increases, then we say that y_n converges to $y(x_n)$

In order that the integrator (2.2) converges, we do require to choose a small meshsize, h. Implementing the linear stiff initial value problem [5]

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} -100 & 0.0025 \\ -1 & -100 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

on the personal computer with the meshsizes, h = 0. 01 and h = 0. 001 respectively.

We observe that the numerical integrator converges to the exact solution in at most 20 iterations for h = 0. 01 and in at most 10 iterations for h = 0. 001.

4. Order Of The Numerical Integrator

LAMBERT [3] and FATUNAL [2] showed that the order of numerical integration could be derived by exploring Taylor series expansion.

According to WALSH [13], the numerical integrator has a truncation error at point $x = x_{n+1}$ $_{n=0,1,2,...}$ and is defined as

$$t_{n+1} = y(x_{n+1}) - y_{n+1} \quad \text{................................} (4.1)$$

where $y(x_{n+1})$ is assumed to be the theoretical solution at $x = x_{n+1}$

Now, the numerical integrator has order p if for problem (1.1.2)

$$t_{n+1} = ch^{p+1} \qquad (4.2)$$

where c is a positive constant. That is, if Taylor series for the theoretical solution $y(x_{n+1})$ and for y_{n+1} coincide up to (and including) the term h^p

we shall determine the order of the numerical integrator, by adopting Taylor expansion of $y(x_{n+1})$ about $x = x_n$ with localizing assumption that there is no previous error, we have

$$t_{n+1} = \left\{ \sum_{i=0}^{\infty} \frac{h^i y^{(i)}}{i!}(x_n) \right\} - \left\{ y_n + G(h)f_n + H(h)f_n^{(1)} \right\}$$

$$= y(x_n) + hy^1(x_n) + \frac{h^2}{2!} y^{(2)}(x_n) + \frac{h^3}{3!} y^{(3)}(x_n) +$$

$$\frac{h^4}{4!} y^{(4)}_{(x_n)} + \frac{h^5}{5!} y^{(5)}_{(x_n)} + \frac{h^6}{6!} y^{(6)}_{(x_n)} - y(x_n)\, G(h)f_n - H(h)f_n^{(1)} + 0(h^7)$$

$$= \frac{h^6}{6!} \left[f^{(5)} - \frac{6!}{h6} \{ G(h)fn + H(h)f_n^{(1)} \} \right] + 0(h^7)$$

hence, the numerical integrator is of order 5 and c = 1/720

5.0. Conclusion

We have discovered that the new numerical integrator proposed by [5] is of order 5 and converges fast especially for a very small mesh size, h.

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