A MATHEMATICAL MODEL OF FOOTBALL PENALTY KICKS

Andrew A. Dosomah
Department of General Studies - Maths Section,
Edo State Institute of Technology and Management, Usen,
Edo State.

Celestina O. Audu
Department of General Studies - Maths Section,
Edo State Institute of Technology and Management, Usen,
Edo State.

And

Maureen E. Oriakhi
Department of Business Administration,
Edo State Institute of Technology and Management, Usen,
Edo State.

Abstract

Football is a popular and lucrative sport. A football team desirous of success should include penalty training as part of match preparation. Penalty training has often been handled from a subjective intuitive perspective of trials and errors and most teams would like to use their best strikers for penalty kicks, however, best strikers sometimes fail to deliver successful penalty kicks. An atmosphere of pressure of high expectation of ovation or threat are typical of penalty kicks. In such atmosphere, maintaining focus, timely performance skills, taking appropriate actions to swerve the goalkeeper or player, knowledge of bound estimates of angular deviation, angular projection and velocity are important factors in the psychology of penalty success. Using mathematics on plain sections of goalkeeper and football dynamics, this research model penalty kicks in total swerving and goalkeeper follow- up in the direction of the ball without ball contact. It sets angular and velocity guides to inform best practices in penalty and also has exciting applications of reality mathematics to football penalties.

Penalties are awarded for serious offences during play. They are also used as decider of victory when a match runs to extra time and ends a draw at the expiration of extra time. Offences that can result in penalty are listed in the laws of the game (FIFA, 2015). There are mathematical models for some aspects of football (Feng and Lu, 2013, Leela and Commissiong, 2009).

Normally, during penalty kicks, a goalkeeper stays at the middle of the post a little distance forward from goal line to maximize the probability of saving the penalty kick. When the referee blows the whistle, the player makes quick skillful movements to swerve the goalkeeper and take the kick while the goalkeeper makes skillful movements at alert to follow the ball. The goalkeeper is totally swerved when he moves to one side of the post while the ball is played to the other side and is a goal or partially swerved when he moves in the same direction as the ball but could not stop the ball from entering the goal. A penalty is saved when a keeper catches the ball before crossing a goal line, deflects the ball away from goal or the player plays the penalty kick outside goal area. Considerations of Dosomah, Audu and Oriakhi for the player are:

- The speed of the ball so that the keeper does not recover from swerving to intercept the ball before it crosses the goal line.
- The side ways angular shift (θ) so that the ball does not go outside the vertical post.
- The angle of projection (α) so that the ball does not become too high and go over the bar.
- Appropriate steps so that the keeper does not easily predict the direction of the ball and enough force to kick the ball in the required direction to satisfy considerations 1 – 3 above.

Review of Some Relevant Projectile Formulars

A football kicked into the air is an example of a projectile. Fig 1 shows the path of a ball kicked from ground into the air and back to the ground due to the force of gravity.

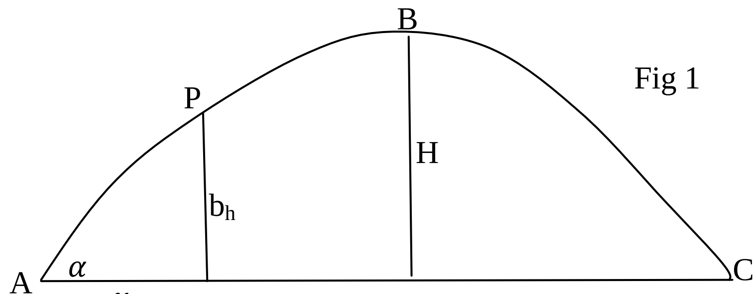


Fig 1

If a ball is kicked from A with an initial velocity (u) at an angle of projection (α) to the ground, all objects thrown in air experience force of gravity which acts downwards, as the ball is going up, the force of gravity will oppose the upward motion and cause the ball to slowdown until it stops at the highest point B. At the highest point, velocity of ball is zero. The force of gravity then drags the ball down increasing its speed as it goes down until it hits the ground at (C). The following formulars are relevant for the dynamics of football penalties in this paper:

- The time (t) from ground to maximum height (B) is given by: $t = \frac{usin\alpha}{g}$
- Neglecting air resistance, time to maximum height equals time from maximum height to ground i.e time from A to B equals time from B to C.
- At any intermediate point (e.g. P) the vertical distance or height of the ball (b_h) is given by: $b_h = utsin\alpha - \frac{1}{2}gt^2$ where t is the time from kick off to P and g is the acceleration due to gravity. The horizontal distance travelled $(x) = utcos\alpha.$ (Halliday, Resnick and Walker, 1997).

Our Model

Allowances for Access

In dynamics, the ball is regarded as a point represented by its centre. When the ball is placed on the penalty spot, the centre of the ball is directly above the spot at distance (r) the radius but the front part of the ball is at a distance (r) from the penalty spot. When the centre of the ball is at the goal line, it is a goal because some part of the ball extends beyond the centre by (r), but when the ball touches the goal post, we often say the ball has reached the goal post in fact, the ball has not, it is the centre of the front part that has reached the post but the centre is at distant (r) before the post because of the solid obstacle. One of the most dangerous penalty is one in which the top of the ball touches the inside bar and the side of the ball touches one the inside of the vertical post. When that happens, the centre of the ball is at a horizontal distance (½w-r) from the midpoint (M) of goal post and is at height (h – r) above the ground, where w is the width of the post and h is the height of the post. Allowing a space of $\frac{1}{20}r = 0.05r$ from the post and top bar for easy access of the ball to goal, we have that the centre of the ball is at a horizontal distance of at most $(\frac{1}{2}w - r - \frac{1}{20}r) = \left(\frac{1}{2}w - 1.05r\right)$ from M and at a height of at most $\left(h - r - \frac{1}{20}r\right) = h - 1.05r$ from ground.

Goalkeeper Stance in Penalty

On goalkeeper stance in penalty, some keepers may squat with hands beside their knees or may spread both hands sideways one to the left and one to the right or may have their hands above knees in prayer mood with eyes alert but no good keeper will likely raise both hands above the head as an initial positional stance to save a penalty.

Derivation of Associated Equations and Simulation Results for Penalties that do Not Attain Maximum Height before Goal Line

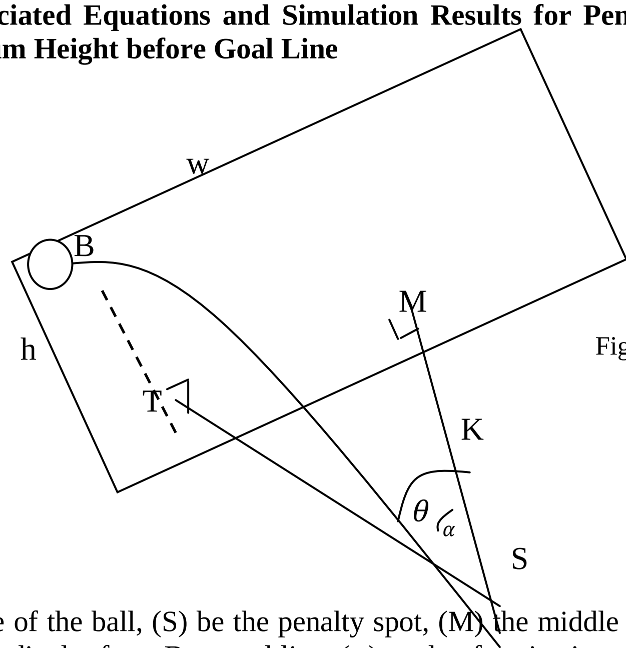


Fig 2

Let (B) be the centre of the ball, (S) be the penalty spot, (M) the middle of the post (T) the foot of the perpendicular from B to goal line, (α) angle of projection of the ball from (S) to B, (θ) the angle of deviation of the ball from line (MS).

Let K be the distance from S to M that is; MS = K (TS) is the horizontal distance travelled by the ball from S to B and (BT) = b_h be the vertical height of ball at B. (Fig 2).

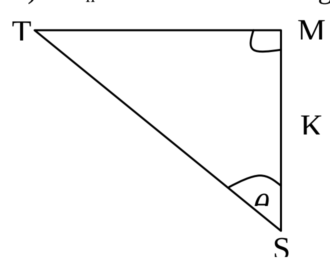


Fig 2a

Using trigonometry on right triangle ΔTMS,

$\frac{MS}{TS} = Cos\ \theta$

$\frac{K}{TS} = Cos\ \theta$

$\frac{MS}{Cos\ \theta} = TS$

$\therefore TS = K\ Sec\ \theta$

From projectile formular, the horizontal distance travelled by the ball from S to B is TS = ut_1 cos α; where (t_1) is the time from S to B.

From equations (1) and (2)

$$Ut_1 cos\ \alpha = k\ sec\ \theta$$

$t_1 = \frac{K\ Sec\ \theta}{U\ Cos\ \alpha} = \frac{K\ Sec\ \theta}{U} \times \frac{1}{\cos\propto}$

$t_1 = \frac{K}{U} sec\theta\ sec \propto$ - - - - - - - (3)

Also $b_h = Ut_1 Sin \propto - \frac{1}{2} g t_1^2$ - - - - - (4)

Substituting for t_I from equation (3) in (4)

We have

$b_h = u \times \frac{k}{u}\ Sec\theta\ sec \propto \times Sin\ \propto\ - \frac{1}{2} g(\frac{k}{u} Sec\ \theta\ Sec\ \propto)^2$

$b_h = K\ Sec\theta tan \propto\ - \frac{1}{2} g \frac{k^2}{u^2} \sec^2 \theta sec^2 \propto$ - - - - (5)

For penalties it is desirable that the ball should move as fast as possible. Thus maximum height should not be attained before or at goal line (since at maximum height, velocity of ball is zero).

$\therefore\ t_1 < \frac{u\ sin\propto}{g}$ - - - - - - - (6)

Substituting t_1 from (3) into (6) gives:

$\frac{k}{u}\ sec\theta\ sec \propto < \frac{u\ sin\propto}{g}$

$\frac{k}{u^2} < \frac{sin\propto}{g\ sec\theta\ sec\propto}$ - - - - - - - (7)

(5) can be written as

$b_n = k\ sec\theta\ tan \propto\ - \frac{1}{2} g \left(\frac{k}{u^2}\right) \times k\ sec^2\theta\ sec^2 \propto$ - - (5a)

Substituting $\frac{k}{u^2}$ from (7) in (5a) gives

$b_n = k\ sec\theta\ tan \propto\ - \frac{1}{2} g \times \left(\frac{sin\propto}{g\ sec\theta\ sec\propto}\right) \times k\ sec^2\theta\ sec^2 \propto$

$b_n = k\ sec\theta\ tan \propto\ - \frac{1}{2}\ ksec\theta\ sec \propto$

$b_n > k\ sec\theta\ tan \propto\ - \frac{1}{2}\ ksec\theta\ tan \propto$

$b_h > KSec\theta \left(tan \propto \times \frac{1}{2} tan \propto\right)$

$b_h > K\ Sec\theta\ \times \frac{1}{2} \tan \propto$

$\therefore k\ Sec\ \theta < \frac{2b_h}{tan\propto}$

$Since\ b_h = h - 1.05r$ for access,

$KSec\theta < \frac{2(h-1.05r)}{tan\propto}$

$tan \propto < \frac{2(h-1.05r)}{Ksec\theta}$ - - - - - - (8)

From (Fig. 2a), the greatest angle of deviation allowing for considerations of access outlined in this paper occurs when $TM = \frac{1}{2} w - 1.05r$

$\therefore tan\theta = \frac{TM}{MS}$

$= \frac{\frac{1}{2} w - 1.05r}{K} = \frac{\frac{1}{2} \times 7.32 - 1.05\ \times 0.11}{11}$

$\tan \theta = 0.322227272$

$\theta = tan^{-1} 0.322227272$

$\theta = 17.86$

Using (8) with $\theta = 17.86, 15, 13, 11,\ 9,\ 7,\ 5,\ 3, 1, 0,\ r = 0.11, k = 11\ gives$;

$\theta = 17.8,\ \propto < 21.91;\ \theta = 15, \propto < 22.20;\ \ \theta = 13, \propto < 22.38;\ \theta = 11, \propto <$ $22.53;\ \theta = 9, \propto < 22.65;\ \theta = 7, \propto < 22.7;\ \theta = 5, \propto = 22.83;\ \theta = 3, \propto < 22.88$

$$\theta = 1, \propto < 22.90; \theta = 0, \propto < 22.91$$

For speed guide,

$$Since\ \frac{K}{U^2} < \frac{Sin\ \propto}{gsec\theta sec\ \propto}$$

$$\frac{kgSec\theta sec\ \propto}{sin\ \propto} < U^2$$

$$\frac{kgsec\theta \times \frac{1}{\cos \propto}}{sin\ \propto} < u^2$$

$$kgsec\theta \times \frac{1}{sin\ \propto\ cos\ \propto} < u^2$$

$$Since\ sin2\ \propto = 2sin\ \propto\ cos\ \propto$$

$$kgsec\theta \times \frac{1}{\frac{1}{2}sin2\ \propto} < u^2$$

$$2kgsec\theta cosec2\ \propto < u^2$$

$$\therefore u = \sqrt{2kgsec\theta cosec2\ \propto} \qquad - \quad - \quad - \quad - \quad - \qquad (9)$$

$Using\ k = 11, g = 9.8, \theta$ $= 17.86, 15, 13, 9, 7, 5, 3, 1, 0\ and corresponding\ values\ of$ $\propto calculated\ from\ (8) in\ (9) gives\ \theta = 17.86, u > 18.09;\ \theta = 15, u > 17.87;\ \theta = 13, u > 17.73;\ \theta = 11, u > 17.62;\ \theta = 9, u > 17.53;\ \theta = 7, u > 17.47;\ \theta = 5, u > 17.40;\ \theta = 3, u > 17.36;\ \theta = 1, u > 17.35;\ \theta = 0, u > 17.34$

To facilitate the calculation, $tan \propto < \frac{4.649cos\theta}{11}$ is used as a simplified form of (8) and $u = \sqrt{215.6sec\theta cosec2\ \propto}$ is used as a simplified form of (9).

Table (1) is a summary of the angular and speed guide.

$\boldsymbol{\theta}$ **(^{0})**	$\propto \leq$**(^{0})**	$\boldsymbol{u} > (\boldsymbol{m}/\boldsymbol{s})$

0	22.91		17.34
1	22.90		17.35
3	22.88		17.36
5	22.83		17.40
7	22.70		17.47
9	22.65		17.53
11	22.53		17.62
13		22.38	17.73
15	22.20		17.87
17.86	21.91		18.09

Safe Range Estimation When Goalkeeper is not Swerved

When a goalkeeper is not swerved, the player should not lose concentration. If the player loses concentration he may miss steps and play at a reduced speed, or play into the hands of the keeper or play off goal. What the player needs is knowledge of safe range estimation to score the goal.

Consider a goalkeeper 2 metres in height, 1 metre arm length and chest width ¾ of arm length and whose legs are ¼ arm length apart. (Fig 3).

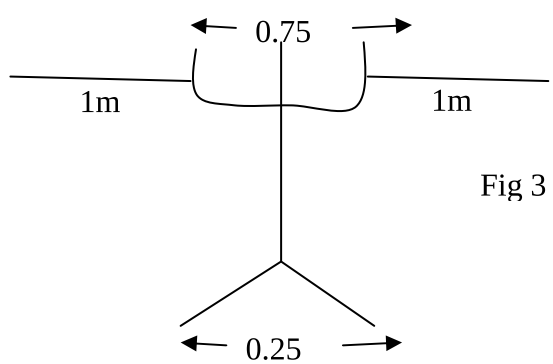
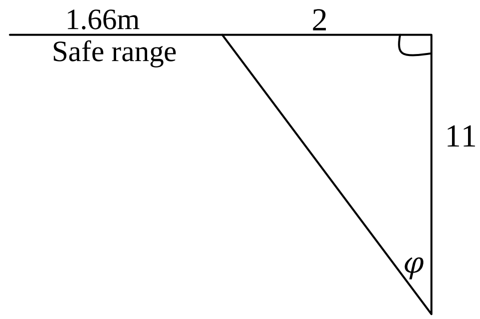


Fig 3

Take stretching as at most ½ arm length. A large estimate of the distance covered from centre $\leq 1m + 0.375m + 0.5m + 0.125m \leq 2m$

$\frac{1}{2} width\ of\ post = \frac{7.32}{2} = 3.66m$



(Fig 4)

Fig 4 depicts the safe range situation from fig 4,

$$\tan\varphi = \frac{2}{11}$$

$\therefore \varphi = 10.30^0$ is greatest deviation of goalkeeper coverage.

Observations

We observe that as the angle of deviation (θ) decreases, the angle of projection required for a given height increases slightly and the minimum required speed decreases slightly. The increase of required projection angles for a given ball height as deviation decreases may be a reason while some players close to goal post, in attempt to play a non-ground ball at small angles of deviation in situations of pressure of opportunity of a directly opposite kick when the keeper is swerved, he may play over the bar because the instinctive tendency to raise angle of projection of a non-ground ball at smaller deviations without angular guide, may be more than required.

Although we have used the greatest feasible ball height and access allowance to derive the formulars, we observed that the formulars are still valid when height is decreased and that a decreased height requires a decreased angle of projection.

Penalty Advice

Angle of deviation (θ) should not exceed 17^0, angle of projection ($\propto$) should not exceed 21^0 and speed should be at least 19m/s. practice playing penalties between 11^0 and 17^0 if the keeper is not swerved, but if swerved, play between 0^0 and 10^0 deviation.

Penalties with Maximum Height Attained before Goal Line

It is undesirable for penalties to attain its maximum height far from goal line. Such penalties will give the keeper a high probability of recovery to catch the ball. The possibility of scoring is when maximum height is attained before goal line. Recall that, time from ground to maximum height is:

$t = \frac{usin\propto}{g}$

To ensure a short time from maximum height to goal line, we add 1% of remaining time to ground to the time to maximum height as the required time for the ball to attain maximum height and descend to goal a short time after the attainment of maximum height.

That is: $t_I = \frac{Usin\propto}{g} + 1\%\ of\ remaining\ time$

$$= \frac{Usin \propto}{g} + \frac{1}{100} \times \frac{Usin \propto}{g}$$

$$= \frac{Usin \propto}{g} (1 + \frac{1}{100})$$

$$= \frac{Usin \propto}{g} (\frac{101}{100})$$

$$If\ t_I = \frac{K}{U} sec\theta sec \propto$$

$$\therefore \frac{101}{100} \frac{Usin \propto}{g} = \frac{K}{U} sec\theta sec \propto$$

$$\frac{101 sin \propto}{100g} = \frac{K}{u^2}\ sec\theta sec \propto$$

Thus,

$$b_h = Ksec\theta tan \propto - \frac{1}{2} g \frac{k^2}{u^2} sec^2\theta sec^2 \propto$$

$$= Ksec\theta tan \propto - \frac{1}{2} gK \times \frac{K}{u^2} sec\theta sec \propto \times\ sec\theta sec \propto$$

$$= Ksec\theta tan \propto - \frac{1}{2} gk \times \left(\frac{101\ sin \propto}{100g}\right) \times sec\theta\ sec \propto$$

$$= Ksec\theta\ tan \propto\ \left(1 - \frac{1}{2} \times \frac{101}{100}\right)$$

$$= Ksec\theta\ tan \propto\ \left(\frac{99}{200}\right)$$

At $b_h = h - 1.05r$, we have $(h - 1.05r) = Ksec\theta\ tan \propto \times\ \frac{99}{200}$

$$\frac{200(h - 1.05r)}{99k\ sec\theta} = tan \propto$$

$$\therefore tan \propto = \frac{200(h - 1.05r)}{99k}\ cos\theta$$

This formular can be simplified as $\tan\theta = \frac{464.9}{99 \times 11} cos\theta$ - - - - (10)

For speed guide,

$$\frac{101}{100}\frac{sin \propto}{g} = \frac{k}{u^2} sec\theta\ sec \propto$$

$u^2 = \frac{100gk\ sec\theta\ sec\propto}{101\ sin\propto}$

$$u^2 = \frac{100gk\ sec\theta}{101\ sin \propto cos \propto}$$

$$u^2 = \frac{100gk\ sec\theta}{101 \times \frac{1}{2} sin2 \propto}$$

$$u^2 = \frac{200gk\ sec\theta\ cosec2 \propto}{101}$$

$$u = \sqrt{\frac{200gKsec\theta\ cosec2 \propto}{101}}$$

The formular for (u) simplifies to $u = \sqrt{213.4653465\ sec\theta\ cosec2 \propto}$

Table 2 shows the result

$\boldsymbol{\theta}$	$\propto \leq$ (⁰)	$\boldsymbol{u > (m/s)}$
0	23.11	17.19
1	23.11	17.19
3	23.08	17.21
5	23.03	17.25
7	22.96	17.30
9	22.86	17.37
11	22.73	17.47
13	22.58	17.57
15	22.40	17.70
17.86	22.11	17.93

Note that for a given height, $b_h = h - 1.05r$, penalties that attain maximum height before goal line require slightly higher angles of projection ($\propto$) and lower minimum speed when compared to penalties that do not attain maximum height before goal line.

Conclusion

This paper is a mathematical model for football penalties in cases of total swerving and goal keeper follow up in ball direction without ball contact. It sets angular and velocity guides to inform best practice in penalties using considerations of allowance for ball access and attainment or non-attainment of maximum height in relevant projectile equations from plain sections of goalkeeper and football dynamics. It gives penalty advice for the player to be effective in penalties for the attention of penalty players and coaches. For non-ground balls the paper recommended penalties that do not attain maximum height before goal line because calculation results show that they have greater speed and lower angles of projection than penalties that attain maximum height before goal line.

Recommendations

Based on the foregoing, the following recommendations are proffered:

1. Practice playing penalties that do not attain maximum height before goal line using the penalty advice and kick as fast as possible.
2. Coaches should use this paper as a training guide for effective penalties.
3. Footballers should study the laws of the game to know of offences that can result in penalties.
4. Footballers should avoid committing penalty offences
5. If granted penalty kicks, penalty players should utilize the opportunities to score goals.

References

FIFA (2015); Laws of the game, Federation International de Football Association, *www.FIFA.com.*

Leela J.K. and Comissiong M.G (2009); Modelling Football penalty kicks, Lat.*Am.J.Phy.Educ*. 3(2) 259 – 269.

Feng Li and Lu L. (2013); Technical Analysis on Mechanical Model Based Football curveball *Research Journal of applied Sciences, Engineering and Technology* 5(14): 3817-3823

Halliday .D, Resnick .R and Walker .J (1997); *Fundamentals of Physics (Volume 1)*, New York, John Wiley & Sons, Inc.