Abstract

In this paper, we consider a non-linear inventory production planning problem with an improved Conjugate Gradient Method. A computer programme was written in Q Basic to apply the Fletcher-Reeves algorithm to the solution of some quadratic problems in production planning. It was then run on Pentium 111 1.2 GHZ computer system. It is proved that the Fletcher-Reeves Conjugate gradient method provides a solution to a non-linear inventory production planning problem.

Introduction

We consider in this paper the inventory cost model

\[ \text{Minimize } C(x, i_1) = C_1(x_1, i_1) + C_2(x_2, i_2) + \ldots + C_n(x_n, i_n) \]

for each period \( t \), the cost incurred depends only on the production quantity \( x_i \) and the ending inventory \( i_i \), and possibly on the period \( t \) itself. We assume that management desires a policy in which the inventory level is zero at the end of the period \( n \) (i.e. \( i_n = 0 \)) and that each period's demand, \( y_t \) must be entirely satisfied on time. This implies

\[ \sum_{i=1}^{n} x_i = y_t, \quad t = 1, 2, 3, \ldots, n \]

Where \( i_0 \) is the specified level of initial inventory at the beginning of the planning horizon, and that entering inventory, \( i_{i-1} \) and production quantity, \( x_i \) must be large enough to make the ending inventory a non-negative quantity. This ensures that the demand is sufficiently satisfied at each period.

Most often, it is expected that the inventory is integer valued, given that demands and production levels are integer valued. Thus if each cost function \( C_i(x_i, i_i) \) is linear then the inventory model is equivalent to a network problem and an appropriate technique of network analysis is employed to solve the problem. In most cases, however, the cost function of an inventory-planning model is non-linear and some techniques of non-linear optimization like dynamic programming, etc. is often applied. In this paper, we shall show that the Fletcher-Reeves conjugate gradient method proves quite suitable for such non-linear inventory cost model.

Fletcher-Reeves Conjugate Gradient Methods

In modern technology, optimization theories are of undisputed importance. The advent of high-speed electronic computers with large memory capacities has given rise fundamentally to the rapid growth of new optimization techniques. And since the main objectives of optimization is to solve a problem under investigation with a high degree of precision and under a highly restrictive operation time, so as to minimize computing cost.

It is necessary to choose a computational scheme that can meet these computational requirements. The desire to construct a suitable and implementable algorithm has motivated the research investigation contained in this paper.

For a brief historical background, we begin, essentially with the outstanding and scholastic publication of Fletcher, R and Reeves C.M (1964). In their joint paper [1] as an answer to the storage difficulty associated with the quasi-Newton methods. The method was first devised by Heslenes and Stiefel (1952). In their paper [2] as a method of solution for linear systems. Fletcher and Reeves built the necessary underlining theory for a successful application of the method to quadratic functional and developed its convergence properties.

They showed that the method has \( n \)-quadratic convergence, that is, it converges in at most \( n \)-iterations starting at any arbitrary point \( x_0 \) [3]. They also have a method of restarting the algorithm so
that the quadratic termination property of the method is maintained, when applied to non-quadratic functional. The Fletcher- Reeves algorithm for executing the CGM is as given below [4].

**Fletcher and Reeves Cgm Algorithm**

The Fletcher- Reeves Conjugate Gradient Method is written as follows:
1. (a) Choose any point $x_0$, compute $g_0$ and set $s_0 = -g_0$
(b) For the $f^{th}$ iteration, $(i \geq 1)$, set the search direction.

\[ s_i = -g_i + \beta_i s_{i-1} \]

Where \[ \beta_i = \frac{\| g_i \|^2}{\| g_{i-1} \|^2} \]

2. Obtain the next point

\[ x_{i+1} = x_i + \lambda^* s_i \]

Where $\lambda^* = \text{Min}_f (x_j + \lambda s_j), \quad \lambda > 0$

This completes the first iteration.

3. Evaluate $g_{i+1}$. If $\| g_{i+1} \|$ is small stop.

Otherwise GO TO 1(b) for the next iteration.

Or GO TO 1(a) if $i = n, 2n, 3n...$ and replace $x_0$ by $x_n, x_{2n}, x_{3n},...$

4. **Remarks**

Any efficient line search technique can be used to evaluate the linear searches

\[ x_{i+1} = x_i + \lambda_i s_i \]

In the case of quadratic functional, Fletcher and Reeves showed that

\[ \lambda_i = \frac{\langle g_i, s_i \rangle}{\langle s_i, Hs_i \rangle} \]

Minimize the line search. Although any efficient method for computing the gradient

\[ g_{i+1} = g(x_{i+1}) \]

can be used, it gives:

\[ g_{i+1} = -\lambda_i Hs_i \]

as an appropriate formula for the gradient of the quadratic functional:

\[ F(X) = \text{Constant} + \langle h, x \rangle + \langle x, Hx \rangle \]

at the point $X_{i+1}$ where $h$, $x$ are vectors and $H$ is the positive definite Hessian Matrix of $F(X)$.

Without restart, the advantage of conjugate directions near the solution point will likely be lost and quadratic termination may not be achieved. The restart procedures help eliminate accumulated errors from the previous iterations so that this does not affect the efficiency of the algorithm when a region is entered in which the function is nearly quadratic. The disappointing slow rate of the method, which was shown to be only linear [5]. Led other contributors to suggest various formulae aimed at improving the algorithm. The suggestion centre on the appropriate formula for $\beta_i$ which modifies the direction vectors $s_i$, Polak, E and Ribierre, R [6]. Proposed that:

\[ \lambda_i = \frac{\langle g_{i+1}, g_{i+1} - g_i \rangle}{\langle g_i, g_i \rangle} \]

And this was shown to have a slight improvement over the Fletcher-Reeves formula. Now, as a restart in the direction of the negative gradient, $-g_i$, generally, result in a smaller reduction of the
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objective function than when the direction \( b \) is taken, Akinyele and Solarin (1999), suggest the formula:

\[
\beta_i = \frac{g_i - g_{i+1}}{S_i + \lambda g_{i+1}}
\]

In the direction:

\[
S_i = g_i + \lambda g_{i+1} + \alpha_i s_i (i \geq 1)
\]

Where:

\[
\alpha_i = \frac{g_i - g_{i+1}}{S_i + \lambda g_{i+1}}
\]

With \( \alpha_i = 0 \), when \( i = 1 \). This formula applied by taking a restart of every \( n \) steps or whenever:

\[
\frac{g_{i+1} - g_i}{\|g_{i+1}\|^2} \geq 0.2 \|g_{i+1}\|^2
\]

was shown by M.D Powell [7], to have a significant improvement on the performance of both the Fletcher-Reeves and Polak-Ribiere algorithms.

**Numeric Implementation**

A computer program was written in Q Basic to apply the Fletcher-Reeves algorithm to the solution of some quadratic problems in production planning. It was then run on Pentium III 1.2 GHZ computer system. Among some test problems used to verify the efficiency of the program is the minimization of the function:

\[
F(X) = 1 + x_1 + x_2 + \frac{1}{4} x_1^2 + x_1 + x_2 + x_2^2 \quad (1)
\]

\[
= 1 + <h, x> + \frac{1}{4} <x, H x>
\]

Where:

\[
h = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}
\]

As expected, the quadratic termination of the method was achieved at not than \( n \) -iterations. Problem (1) above was minimized at:

\[
X^* = \begin{pmatrix} 0.000000894 \\ -1 \end{pmatrix}
\]

With the minimum value \( F(X^*) = 0.5 \) achieved after 2 iterations.

**Application to Production Planning**

We consider applying the Fletcher Reeves CGM to the industrial problem of production planning. In this inventory problem, the company produces different types of products, we shall concentrate on the production and sales of a particular type of product of which the lead-time is known and is constant.
Our Industrial Model

Skylight Limit Agricultural Engineering Company, Benin City, Nig. Ltd., specializes in (lie manufacturing of agricultural tools and farming implements and other agricultural product processing machines such as Cassava grinding machines, Palm Kernel cracking Machines, Cutlasses, Shovels, Wheelbarrows and Sickles etc. From its past records, the company realized that in the first quarter of the year it usually, receives orders to supply an average of 100 units of sickles per month. Now the cost of producing \( x \) units of sickle (s) in any month is \( Nx^2 \), and the company is currently faced with (he problem of determining (he level of production to undertake this quarter in order to minimize costs; though (he company is capable of producing more units of sickles than it supplies per month and carries the remnants to a subsequent month, but an inventory holding cost of NJO per unit sickle will be incurred in a month. Moreover, the company does not charge ordering cost on any of its transactions. So what should be done?

Mathematical Formulation and Solution

For the mathematical formulation of (he problem, we shall assume that there is no initial inventory since we have no information on the correct inventory level at the time of writing this paper.

Let \( x_1, x_2 \) and \( x_3 \) represent the unit of sickle products in the first, second and third months respectively; and let the total cost to be minimized be given as: \( F(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + 10(x_1 - 100) + 10(x_2 + x_3 - 200) \) Where Total inventory Cost=(Production Cost+Holding Cost) The Mathematical Model is: Min. \( F(X)=x_1^2+x_2^2+x_3^2+20x_1+20x_2+3000 \) (1)

This is an inequality constrained minimization problem to which we apply the Kuhn Tucker conditions. However, the equivalent unconstrained form is given by the Langrange equation \( L \).

\[
\min L(X,\lambda)=\sum_{i=1}^{3} x_i^2+20x_1+10x_2+3000+\lambda_1(x_1-100)+\lambda_2(x_2+200)+\lambda_3(x_3+300) \quad (2)
\]

Where \( \lambda_i = 0 \quad i = 1, 2, 3 \)

The Kuhn-Tucker conditions can be stated as:

\[
\frac{\partial F}{\partial x_i} + \lambda_i \frac{\partial g_i}{\partial x_i} = 0 \quad \text{where } i = 1, 2, 3
\]

\( \frac{\partial F}{\partial x_i} = 2x_i + 20 + \lambda_1 + \lambda_2 + \lambda_3 \quad \text{for } i=1,2,3 \)

\( \lambda_2(x_1 + x_2 - 200) = 0 \quad (6) \)

\( \lambda_3(x_1 + x_2 + x_3 - 300) = 0 \quad (7) \)

Therefore:

\( x_1 - 100 \geq 0 \)

Subject to:
d. $X_j$ 0, since it is a minimization problem: implies
$$\begin{align*}
X_1 & = 0 \\
X_2 & = 0 \\
X_3 & = 0
\end{align*}$$

We have $X_p$ -10, $X_2$ -10 and $X_3$ -200

Since we are interested in the global minimum of $X_j$

Global Min $(X_j) = 200$

Thus from equation (2) the required unconstrained form of the problem is:
$$L(X) - X_j^2 - 4X_j^2 - 1000X_j + 60000$$

**Application of Fletcher Reeves CGM to the Problem**

Using the values obtained above from the Kuhn-Tucker multipliers $\lambda_1, \lambda_2, \lambda_3$, we see that the resultant unconstrained problem:

$$L(X) = x_1^2 + x_2^2 + x_3^2 - 1000(x_1 + x_2 + x_3) + 60000$$

is a quadratic functional in $x_1, x_2, x_3$. Hence, we can apply the Fletcher Reeves CGM algorithm as follows:

Take $X_0 = (0, 0, 0)^T$

Let $g_0 = \nabla L(X_0)$

But:

$$S_0 = -g_0 = (200, 200, 200)^T$$

Hence the direction of search from $X_0$ is:

$$S_0 = -g_0 = (200, 200, 200)^T$$

Now $\nabla^2 L(X) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

so, we take a step length

$$\lambda_0 = \frac{\langle g_0, S_0 \rangle}{\langle S_0, H S_0 \rangle} = \frac{1}{2}$$

Where $H = \nabla^2 L(X)$

Thus $\lambda$ minimizes the line search at $X_i = X_0 + \lambda_0 S_0 = (100, 100, 100)^T$

As a result, we find the gradient $g_i = g_0 + \lambda_0 HS_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Implies $g_i = 0$, this means that the Fletcher Reeves CGM converges after one iteration!

Hence the convergence point of the problem is at:

$$X^* = (100, 100, 100)$$

which shows that the company should produce only 100 units of sickles in the first month, 100 units of sickles in the second month and in the third month, so its cost of production will be at a minimum,
We have $X_1 = -10$, $X_2 = -10$ and $X_3 = -200$

Since we are interested in the global minimum of $X_j$

Global Min $X_i = -200$

Thus from equation (2) the required unconstrained form of the problem is: $L(X) = x_1^2 + 2x_2 + 2x_3 - 200(x_1 - x_2 - x_3) + 60000$

Application of Fletcher Reeves CGM to the Problem

Using the values obtained above from the Kuhn-Tucker multipliers $X_1, X_2, X_3$, we see that the resultant unconstrained problem:

$L(X) = x_1^2 + x_2^2 + x_3^2 - 200(x_1 - x_2 - x_3) + 60000$ is a quadratic functional in $X_1, X_2, X_3$

Hence, we can apply the Fletcher Reeves CGM algorithm as follows:

Take $X_0 = (0, 0, 0)^T$

go = $-V(L(X_0))$

But: $So = -go$

Hence the direction of search from $X_0$ is:

$S_{u} = -g_{r}(200, 200, 200)^T$

Now $V^2L(X) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

so, we take a step length

$X_0 = \langle go, go \rangle = \frac{r}{\sqrt{2}}$

Where $II - V^2L(X)$

Thus $X_i$ minimizes the line search at $X_i = X_0 + X_0 S_0 = (100, 100, 100)^T$

As a result, we find the gradient $g_i = go + X_0 S_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Implies $g_i = 0$, this means that the Fletcher Reeves CGM converges after one iteration! Hence the convergence point of the problem is at:

$X^* = (100, 100, 100)$

which shows (hat the company should produce only 100 units of sickles in the first month, 100 units of sickles in the second month and in the third month, so its cost of production will be at a minimum,
estimated to be N60000.00 (Sixty thousand Naira) for the production of sickles during the season and directs its other resources to the production of other products like Palm kernel cracking machines, Cutlasses, Shovels, etc.

**Conclusion**

The high-speed convergence rate of the Fletcher Reeves conjugate gradient method provides a solution to a non-linear inventory production planning problem.

**References**


