A NEW CONVERGENT SERIES SOLUTION OF THE BLOCH NUCLEAR MAGNETIC RESONANCE FLOW EQUATIONS

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Abstract

The importance of magnetic resonance in classical diagnoses and prognoses cannot be over emphasized. Among its wide clinical applications is that it provides accurate information, which is of a physiological and morphological nature at molecular level, with respect to the metabolic processes in the human body. It is also useful for the quantification of the blood flow pattern in the system of human body. This paper presents the convergent series solution of the Bloch nuclear magnetic resonance equations for magnetic resonance imaging with time varying rF B_1Field.The solution can be used to extract information in the velocities of interest from the measured MR phase modulated signal.

Introduction

Magnetic resonance is a phenomena produced by simultaneously applying a steady magnetic field and electro magnetic radiation (usually radio waves) to a sample of atoms and then adjusting the frequency of the radiation and the strength of the magnetic field to produce absorption of the radiation(Boer,1996).

Nuclear magnetic resonance simply refers to resonance of protons to radiation in a magnetic field. Nuclear magnetic resonance is analogous to Electron Paramagnetic Resonance (EPR), however NMR is produced by the much smaller magnetism associated with impaired nuclear spins. The NMR resonant frequency (usually that of protons in complex molecules) is slightly shifted by interaction with nearby atoms in the sample, thus providing information about the chemical structure of organic molecules and other materials. NMR is now extensively employed in medicine, although the use of the world “nuclear” is avoided, the preferred name being magnetic resonance imaging. The technique provides high-quality cross- sectional images of internal organs and structures. The imaging sequence can be modified to visualize blood flow to compensate for the blurring effects of cardiac or respirator motion(Makinde,2006). Magnetic resonance also offers the unique ability to acquire images in virtually any direction, without repositioning the patient. This translates into greater convenience for medical staff and minimized patient discomfort. Furthermore, magnetic resonance provides chemical information not measurable with conventional radiography or ultrasonography. It is the combination of its versatility, sensitivity and specificity as a diagnostic modality that has accelerated the acceptance of magnetic resonance imaging.

Though, there have been tremendous theoretical and practical advent improvements of these MR flow techniques, new results of intensive current researches in this filed are continuously being presented at scientific meetings. However, a great deal of further research is needed to exhaust all the quantitative information for studying hemodynanics and study dynamics by magnetic resonance. An ideal approach to this further research would be to find generalized time dependent analytical solution to the Bloch NMR flow equations (Archie,1991).

The Bloch equations are coupled non linear equation describing the motion of a macroscopic magnetization 'm' under the influence of applied magnetic fields. There are no simple closed solutions known for a general rF excitation. Therefore generalized analytical solution of the Bloch NMR flow equation is obviously a very difficult task.

Awojoyogbe and Salako(2000) used Picard's method to obtain the solution of Bloch Nuclear Magnetic Resonance flow Equations the results obtained were accurate and stable, but it is bedeviled by some computational difficulties which make them impracticable. These difficulties include

(i) cumbersome derivation process

(ii) large amount of functions evaluation

(iii) requirement of large amount of time

The method, which we are considering in this paper, reduces the amount of difficulties and makes it more general in application than present. The modified method is new convergent series solution of Bloch Equations for NMR Flow with spatially varying magnetic field gradient.

In this contribution, a convergent series solution of Bloch equations for NMR. Fluid flow with spatially varying magnetic field gradient is presented.
For this investigation, we assumed that resonance condition exist at larmor frequency Awojoyogbe(1999)

$$f_0 = \gamma B - \omega = 0 \qquad (2.1)$$

γ is the gyro magnetic ratio of blood spins: $\frac{\omega}{2\pi}$ is the rF excitation frequency; f_0/γ is the off resonance filed in the rotating frame of reference. The x,y,z components (in the rotating frame) of magnetization of a fluid bolus is given by the Bloch equation, which may be written as follows.

$$\frac{dMx}{dt} = V * grad\ M_X + \frac{\partial M_X}{\partial t} = -\frac{M_X}{T_2}$$

$$\frac{dM_y}{dt} = V * \text{grad}\, M_y + \frac{\partial M_y}{\partial t} = \gamma M_x B_1(x) - \frac{M_y}{T_2}$$

$$\frac{dM_z}{dt} = V * \text{grad}\, M_z + \frac{\partial M_z}{\partial t} = \gamma M_x B_1(x) - \frac{M_0 - M_z}{T_1}$$

Two reasonable initial boundary conditions, which may conform to the real-time experimental arrangements, are chosen. These are

1. $M_o \neq M_Z$

a situation, which holds good in general and in particular when the rF B1 (x) field is strong, say of the order of 1.0 G or more so that M_z of the fluid bolus changes appreciably from M_o.

2. Before entering the signal detection system, blood bolus has magnetization $M_x = 0$, $M_y = 0$ (Rien,1991)

For steady flow $\frac{\partial M_y}{\partial t} = 0$

From equations (2.2) and (2.3) we obtained:

$$\frac{d^2 M_y}{dx^2} + \frac{1}{V}\left[1/T_1 + 1/T_2\right]\frac{dM_y}{dx} + \frac{\gamma^2 B^2{}_1}{V_2}(x) M_y = \frac{\gamma M_0 B_1(x)}{V_2 T_1}$$

$$Setting\ R = \frac{1}{V}\left[1/T_1 + 1/T_2\right];\quad S(x) = \frac{\gamma^2}{V^2} B^2{}_1(x) = PB_1^2(x) \quad and \quad Q(x) = \frac{\gamma M_0}{V^2 T_1} B_1(x) = \phi B_1(x)$$

Thus equation (2.5) becomes:

$$\frac{d^2 M_y}{d^2 x^2} + \frac{R dM_y}{dx} + PB_1^2(x) M_y = \varphi B_1(x)$$

$$\text{Where } P = \frac{\gamma^2}{V_2};\quad \varphi = \frac{\gamma M_0}{V_2 T_1} \quad \Rightarrow \text{Velocity dependent terms.}$$

Assume B_1 (x) is a linear function of the form:

$B_1 (x) = ax + b$ (2.7)

$B_1{}^2(x) = a^2x^2 + 2abx + b^2$ (2.8)

Setting $A = a^2$; $k = 2ab$ and $c = b^2$

Thus equation (2.6) becomes

$$\frac{d^2M_y}{d^2x^2} + \frac{RdM_y}{dx} + P(Ax^2 + kx + c)M_y = \varphi\ (ax+b) \quad (2.9)$$

The point $x = x_0$ is an ordinary point. Thus we have that:

$$M_y^{\Pi} + RM_y^{I} + P(Ax^2 + kx + c)M_y = \varphi ax + \varphi b \quad (2.10)$$

Assuming a solution of the form:

$$M_y = \sum_{r=0}^{\infty} \alpha_r x^r \quad \text{(Makinde,2000)} \qquad (2.11a)$$

$$i.e.\ M_y^{\ 1} = \sum_{r=1}^{\infty} r\alpha_1 x^{r-1} \qquad (2.11b)$$

and

$$M_y^{\ ii} = \sum_{r=2}^{\infty} r(r-1)\alpha_r x^{r-2} \qquad (2.11c)$$

Substituting equations (2.11:a,b, c0 into (2.10) to have:

(2.12)

$$\sum_{r=2}^{\infty} r(r-1)\alpha_r x^{r-2} + R\sum_{r=1}^{\infty} r\alpha_r x^{r-1} + P\left\{Ax^2 \sum_{r=0}^{\infty}\alpha_r x^r + kx\sum_{r=0}^{\infty}\alpha_r x^r + c\sum_{r=0}^{\infty}\alpha_r x^r\right\} = \varphi ax + \varphi b$$

(2.13)

$$\sum_{r=2}^{\infty} r(r-1)\alpha_r x^{r-2} + R\sum_{r=1}^{\infty} r\alpha_r x^{r-1} + P\left\{A\sum_{r=0}^{\infty}\alpha_r x^{r-2} + k\sum_{r=0}^{\infty}\alpha_r x^{r-1} + c\sum_{r=0}^{\infty}\alpha_r x^r\right\} = \varphi ax + \varphi b$$

(2.14)

Setting r = r-1, r = r-2, r = r-3 and r = r-4 in the 2nd ,3rd, 4th and 5th terms respectively. We then have that:

$$\sum_{r=2}^{\infty} r(r-1)\alpha_r x^{r-2} + R\sum_{r=2}^{\infty}(r-1)\alpha_{r-1}x^{r-2} + PA\sum_{r=4}^{\infty}\alpha_{r-4}x^{r-2} + Pk\sum_{r=3}^{\infty}\alpha_{r-3}x^{r-2} + Pc\sum_{r=2}^{\infty}\alpha_{r-2}x^{r-2} = \varphi ax + \varphi b$$

2.15

Taking two terms in the 1st , 2nd , and the last summations respectively: and one term in the 4th summation.

$$2\alpha_2 + 6\alpha_3 x) + \sum_{r=4}^{\infty} r(r-1)\alpha_r x^{r-2} + R(\alpha_1 + 1\alpha_2 x) + R\sum_{r=4}^{\infty}(r-1)\alpha_{r-1}x^{r-2} + PA\sum_{r=4}^{\infty}\alpha_{r-4}x^{r-2}$$

$$+ Pk\alpha_0 x + Pk\sum_{r=4}^{\infty}\alpha_{r-3}x^{r-2} + Pc(\alpha_0 + \alpha_1 x) + Pc\sum_{r=4}^{\infty}\alpha_{r-2}x^{r-2} = \varphi ax + \varphi b$$

(2.16)

Combining the like terms together, we have:

$$2\alpha_2 + R\alpha + Pc\alpha_0 - \varphi b) + (6\alpha_3 + 2R\alpha_2 + Pk\alpha_0 + Pc\alpha_1 - \varphi a)x$$

$$+ \sum_{r=4}^{\infty}\left\{r(r-1)\alpha_r + R(r-1)\alpha_{r-1} + PA\alpha_{r-4} + Pk\alpha_{r-3} + Pc\alpha_{r-2}\right\}x^{r-2} = 0$$

(2.17)

2.1 Recurrence Formula

With coefficients vanishing identically

i.e. $2\alpha_2 + R\alpha_1 + Pc\alpha_0 - \varphi b = 0$

(2.18a)

$$6\alpha_3 + 2R\alpha_2 + Pk\alpha_0 + Pc\alpha_1 - \varphi a = 0$$
(2.18b)

$$r(r-1)\alpha_r + R(r-1)\alpha_{r-1} + PA\alpha_{r-4} + Pk\alpha_{r-3} + Pc\alpha_{r-2} = 0$$
(2.18c)

From equation (2.18a) we have that:

$$\frac{\alpha_2 = \varphi b - (R\alpha_1 + Pc\alpha_0)}{2} \tag{2.19}$$

Similarly from (2.18b) we have also:

$$\frac{\alpha_3 = \varphi a - (2R\alpha_2 + Pk\alpha_0 + Pc\alpha_1)}{6}$$

Thus substituting $\alpha_2 \operatorname{int} o \alpha_3$ we have:

$$\frac{\alpha_3 = \varphi(a - Rb) + P(Rc - k)\alpha_0 + (R^2 - Pc)\alpha_1}{6} \tag{2.20}$$

And also from (2.18c) we have that:

$$\frac{\alpha_r = -\{PA\alpha_{r-4} + Pk\alpha_{r-3} + Pc\alpha_{r-2} + R(r-1)\alpha_{r-1}\}}{r(r-1)} \tag{2.21}$$

Equation (2.21) Is the Recurrence Formula of α_R Bloch function.
Hence the solution of equation (2.6) is given as:

$$M_y = \sum_{r=0}^{\infty} \alpha_r x^r$$

$$= \sum_{r=0}^{\infty} -\frac{\{PA\alpha_{r-4} + +Pk\alpha_{r-3} + Pc\alpha_{r-2} + R(r-1)\alpha_{r-1}\}x^r}{r(r-1)} \tag{2.22}$$

From equation (2.21), r takes value from 4, we have the following expressions:

$$\frac{\alpha_4 = -\{PA\alpha_0 + Pk\alpha_1 + Pc\alpha_2 + 3R\alpha_{r3}\}}{12} \tag{2.23}$$

$$\frac{\alpha_5 = -\{PA\alpha_1 + Pk\alpha_2 + Pc\alpha_3 + 4R\alpha_4\}}{20} \tag{2.24}$$

$$\frac{\alpha_6 = -\{PA\alpha_2 + Pk\alpha_3 + Pc\alpha_4 + 5R\alpha_5}{30} \tag{2.25}$$

$$\frac{\alpha_7 = -\{PA\alpha_3 + Pk\alpha_4 + Pc\alpha_5 + 6R\alpha_6\}}{42} \tag{2.26}$$

$$\frac{\alpha_8 = -\{PA\alpha_4 + Pk\alpha_5 + Pc\alpha_6 + 7R\alpha_7\}}{56} \tag{2.27}$$

$$\frac{\alpha_9 = -\{PA\alpha_5 + Pk\alpha_6 + Pc\alpha_7 + 8R\alpha_8\}}{72} \tag{2.28}$$

Substituting equation (2.19),(2.20)into(2.23) to have:

$$12\alpha_4 = -\frac{\{PA\alpha_0 + Pk\alpha_1 + Pc[\varphi b - (R\alpha_1 + Pc\alpha_0) + R[\varphi(a - Rb) + P(Rc - k)\alpha_o + (R^2 - Pc)\alpha_1 \}]]}{2}$$

Combination the like terms to have

$$\alpha_4 = -\frac{\{\varphi[Pcb + R(a - Rb\}] + [PR(Rc - k) + P(2A - Pc^2)]\alpha_o + [R(R^2 - 2Pc) + 2PK]\alpha_1\}]]\}}{24}$$

(2.29)

Putting Equations (2.19),(2.20)and (2.29)into (2.24)

$$20\alpha_5 = -\{\frac{PA\alpha_1 + Pk/2[\varphi b - (R\alpha_1 + Pc\alpha_0)] + Pc/6[\varphi(a - Rb) + P(Rc - k)\alpha_0 + (R^2 - Pc)\alpha_1]}{6}$$

$$\frac{-R[\varphi[Pcb + R(A - Rb)] + [PR(Rc - k) + P(2A - Pc^2)]\alpha_0 + [R(R^2 - 2Pc) + 2PK]\alpha_1] \}}{6}$$

$$\alpha_5 = -$$

$$\{\varphi[3Pkb + Pc(a - 2Rb) - R^2(a - Rb)] + [2P^2c(Rc - k) - PR^2(Rk + c) - 2RPA]\alpha_0 +$$

$$[6PA - 5PkR - R^4 - PcR^2 - P^2c^2]\alpha_1\}/120$$

(2.30)

Similarly: putting (2.19), (2.20), (2.29) and (2.30) into (2.5)

$$30\alpha_6 = -\{\frac{PA/2[\varphi b - (R\alpha_1 + Pc\alpha_0)] + Pk/6[\varphi(a - Rb) + P(Rc - k)\alpha_0 + (R^2 - Pc)\alpha_1]^1}{6}$$

$$\frac{-R/24([\varphi[3Pkb + Pc(a - 2Rb) - R^2(a - Rb)] + [2P^2c(Rc - k) - PR^2(RK + c) - 2RPA]\alpha_0}{6}$$

$$+[6PA - 5PkR - R^4 - PcR^2 - P^2c^2]\alpha_1) - Pc/24(\varphi[Pcb + R(a - Rb)] + [PR(Rc - k)$$

$$+P(2A - Pc^2)\alpha_0] + [R(R^2 - 2Pc) + 2Pk]\alpha_1) \}]$$

Combining like terms to have

$$720\alpha_6 = -\{\varphi[12Pab + 4PK(a - Rb) - Pc[Pcb + R(a - Rb)] - R][3kPb + Pc(a - 2Rb)$$

$$-R^2(a - Rb)]] + [12P^2Ac + 4P^2k(Rc - k) - Pc[PR(Rc - k) + P(2A - Pc^2)$$

$$-R[2P^2c(Rc - k) - PR^2(Rk + c) - 2PAR]]\alpha_0 - [1 - 12PAR + 4Pk(R^2 - Pc)$$

$$-Pc[R(R^2 - 2Pc) + 2Pk -]R[6PA - 5PkR - R^4 - P^2c^2 - PcR^2]]\alpha_1 \}$$

Hence we have:

$$.:\alpha_6 = -\{\varphi[12Pab + Pk(4a - 7Rb) - Pc(Pcb + 2Ra) + 3R^2bPc + R^3(a - Rb)\ ]$$

$$+ 7P^2c(2A + kR) + 2PR^2a - 4P^2k^2 - 3P^2k^2R^2 + P^3c^3 + PR^4k + PR^3c\ ]\alpha_0$$

$$+ [9PkR^2 - 18PAR - 6P^2kc + R^5 + P^2c^2(2 + R)\ ]\alpha_1\ \}/720$$

Thus equation (2.22) becomes:

$$M_y = M_{y_1} + M_{y_2} + M_{y_3}$$

Where:

$$M_y = \varphi\left\{\frac{bx^2}{2} + (\frac{a - Rb}{6})x^3 - (\frac{Pcb + R(a - Rb))x^4}{24} + \left[3kP\frac{b + Pc(a - Rb) - R^2(a - rb)\ ]}{120}x^5 + --- \right.\right\}$$

$$M_{y_2} = -\left\{\frac{Pc^2x^2}{2} - \frac{P(Rc - k\ )x^3}{6} + \frac{[PR(Rc - k) + P(2A - Pc^2)x^4]}{24}\right.$$

$$+ \frac{[2P^2c(Rc - k) - PR^2(Rk + c) - 2RPAx^5 + -----]}{120}\alpha_0$$

$$M_{y_3} = -\frac{\{Rx^2}{2} - \frac{(R^2 - Pc)x^3}{6} + \left[R(\frac{R^2 - 2Pc) + 2Pk}{24}x^4 + \left[6PA - 5P\frac{kR - R^4 - PcR^3 - P^2c^2}{120}x^5\right]\alpha_1 + --- \right.\right\}$$

M_{y2} and M_{y3} can be confidently neglected, because they do not contain the term M_0 where φ thus.

Also by neglecting all other terms embedded in term φ . We thus have the below expression:

$$M_y = M_{y1} = \varphi\frac{\{bx^2}{2} + \frac{(a - Rb)x^3}{6} - \frac{R(A - Rb)x^4}{24} + R^2(\frac{A - rB)\ ]}{120}\frac{x^5}{} - R\frac{^3(A - Rb)x^6}{720} - -\ \}$$

(2.31)

From equation (2.31) above, we have the general formula as

$$M_y = \varphi\sum_{n=2}^{\infty}(-1)^n R^{(n-2)}(\frac{ax^{n+1}}{(n+1)!} + \frac{bx^n)}{n!}$$

(2.32)

For b = 0 we have that:

$$M_y = \varphi\left\{\frac{ax^3}{3!} - \frac{Rax^4}{4!} + \frac{R^2ax^5}{5!} - \frac{R^3ax^6}{6!} + \frac{R^4aX^7}{7!} - \right\}$$

$$M_y = a\varphi\sum_{n=2}^{\infty}(-1)^n R^{(n-2)}\frac{X^{n+1}}{(n+1)!}$$

(2.33)

Conclusion

A cursory look at the derived solution shows that this method is fast and more direct than the Picard's method already used by previous researchers. The new convergent series method is particularly attractive because it does not require large amount of functions evaluation and therefore could be used as formula in a multipurpose code for analysis of real life problems arising in medical physics, medicine and other applied sciences.

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